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Continuation Passing Style for Effect Handlers

Formal Structures for Computation and Deduction

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Algebraic effects by example: a drunk coin toss

Consider two effects: nondeterminism and exceptions

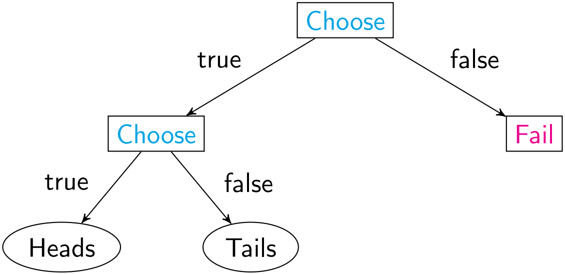
$$
\begin{aligned}
&\mathsf{drunkToss} : \mathsf{Toss} \mathrel{!} \{\mathsf{Choose} : \mathsf{Bool}; \mathsf{Fail} : \mathsf{Zero}\} \\
&\mathsf{drunkToss} = \textbf{if do } \mathsf{Choose} \textbf{ then} \\
&\qquad\qquad\qquad \textbf{if do } \mathsf{Choose} \textbf{ then } \mathsf{Heads} \textbf{ else } \mathsf{Tails} \\
&\qquad\qquad\quad \textbf{else absurd do } \mathsf{Fail}
\end{aligned}
$$

Algebraic effects by example: a drunk coin toss

Consider two effects: nondeterminism and exceptions

drunkToss : Toss ! {Choose : Bool; Fail : Zero}
drunkToss = **if do** Choose **then**
    if do Choose **then** Heads **else** Tails
  else absurd do Fail

Drunk toss computation tree



Effect handlers by example: modular interpretations

Choose handler

$$\mathsf{allChoices} : \alpha \; ! \; \{\mathsf{Choose} : \mathsf{Bool}; \rho\} \Rightarrow \mathsf{List}\ \alpha \; ! \; \{\rho\}$$
$$\begin{aligned}\mathsf{allChoices} = \;& \mathbf{return}\ x \mapsto [x] \\ & \mathsf{Choose}\ r \mapsto r\ \mathsf{true} \mathbin{+\!\!+} r\ \mathsf{false}\end{aligned}$$

Fail handler

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$$\begin{aligned}\text{fail} = \ & \textbf{return } x \mapsto [x] \\ & \text{Fail } r \mapsto []\end{aligned}$$

Two possible interpretations:

$$\begin{aligned}& \textbf{handle } (\textbf{handle } \text{drunkToss } \textbf{with } \text{fail}) \textbf{ with } \text{allChoices} \\ & \Rightarrow [[\text{Heads}], [\text{Tails}], []]\end{aligned}$$

$$\begin{aligned}& \textbf{handle } (\textbf{handle } \text{drunkToss } \textbf{with } \text{allChoices}) \textbf{ with } \text{fail} \\ & \Rightarrow []\end{aligned}$$

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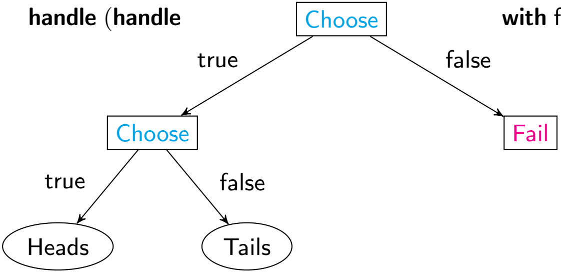
Two possible interpretations:

handle (**handle**



with fail) **with** allChoices

Effect handlers by example: modular interpretations

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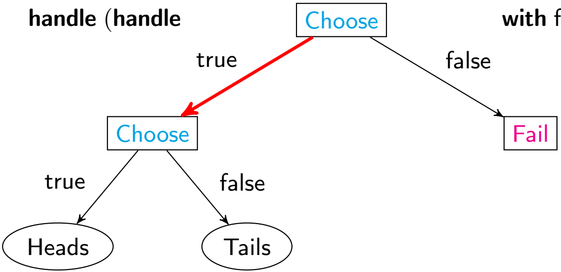

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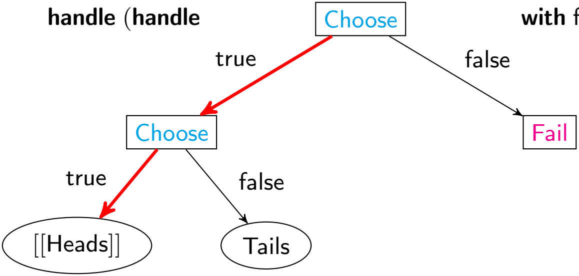
handle (**handle**  **with** fail) **with** allChoices

Effect handlers by example: modular interpretations

Choose handler

allChoices : α ! {Choose : Bool; ρ} $\Rightarrow$ List α ! {ρ}
allChoices = **return** $x \mapsto [x]$
Choose $r \mapsto r$ true ++ r false

Fail handler

fail : α ! {Fail : Zero; ρ} $\Rightarrow$ List α ! {ρ}
fail = **return** $x \mapsto [x]$
Fail $r \mapsto []$

Two possible interpretations:

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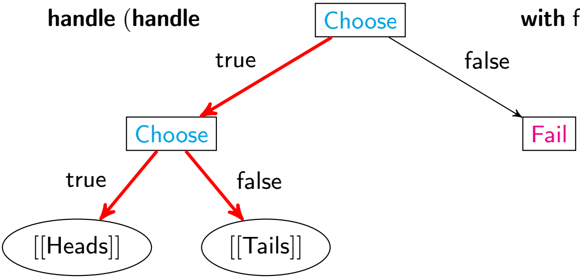

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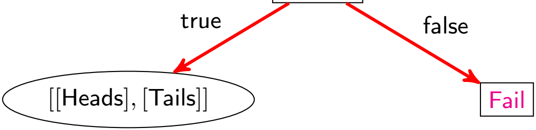

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fail : α ! {Fail : Zero; ρ} $\Rightarrow$ List α ! {ρ}
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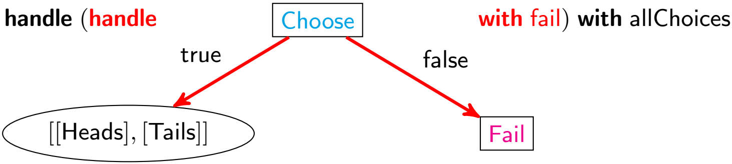
Fail $r \mapsto$ []

Two possible interpretations:



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$$\begin{array}{l}\mathsf{allChoices} : \alpha \mathbin{!} \{\mathsf{Choose} : \mathsf{Bool}; \rho\} \Rightarrow \mathsf{List}\ \alpha \mathbin{!} \{\rho\} \\ \mathsf{allChoices} = \mathbf{return}\ x \mapsto [x] \\ \qquad\qquad\quad \mathsf{Choose}\ r \mapsto r\ \mathsf{true} \mathbin{+\!\!+} r\ \mathsf{false}\end{array}$$

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Two possible interpretations:

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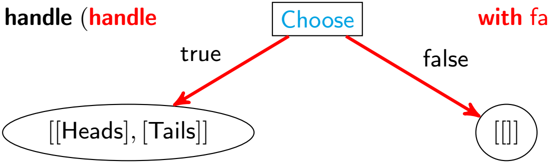

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Two possible interpretations:

[[Heads], [Tails], []]

Effect handlers by example: modular interpretations

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$$
\begin{aligned}
&\mathbf{handle}\ (\mathbf{handle}\ \mathsf{drunkToss}\ \mathbf{with}\ \mathsf{allChoices})\ \mathbf{with}\ \mathsf{fail} \\
&\Rightarrow []
\end{aligned}
$$

Effect handlers by example: modular interpretations

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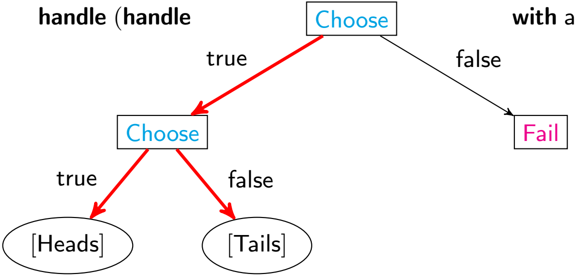

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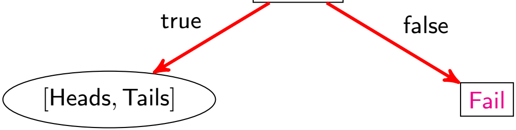

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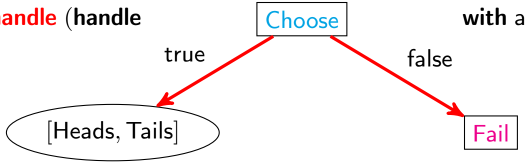

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Two possible interpretations:

$[]$

Operational semantics for effect handlers

The operational semantics for handlers is fairly simple

$$\begin{array}{l}\textbf{handle}\ (\textbf{return}\ V)\ \textbf{with}\ H \rightsquigarrow N[V/x],\\ \qquad \text{where } H^{\mathrm{ret}} = \{\textbf{return}\ x \mapsto N\}\end{array}$$

$$\begin{array}{l}\textbf{handle}\ \mathcal{E}[\textbf{do}\ \ell\ V]\ \textbf{with}\ H \rightsquigarrow N[V/p, \lambda y.\,\textbf{handle}\ \mathcal{E}[\textbf{return}\ y]\ \textbf{with}\ H/r],\\ \qquad \text{where } \ell \notin BL(\mathcal{E}) \text{ and } H^{\ell} = \{\ell\ p\ r \mapsto N\}\end{array}$$

The set of bound labels, BL, is defined inductively

$$BL([\ ]) = \emptyset \quad BL(\textbf{let}\ x \leftarrow \mathcal{E}\ \textbf{in}\ N) = BL(\mathcal{E}) \quad BL(\textbf{handle}\ \mathcal{E}\ \textbf{with}\ H) = BL(\mathcal{E}) \cup \mathrm{dom}(H)$$

Evaluation contexts, $\mathcal{E}$, are standard

$$\mathcal{E} ::= [] \mid \textbf{let}\ x \leftarrow \mathcal{E}\ \textbf{in}\ N \mid \textbf{handle}\ \mathcal{E}\ \textbf{with}\ H$$

Why continuation passing style?

So, effect handlers are great! Now, how do we compile them?

Why continuation passing style (CPS)?

- CPS is good for implementing control flow
- CPS is formulated on a restricted subset of λ-calculus
- The Links compiler (Cooper, Lindley, Wadler, and Yallop 2006)
  - Server-side uses a CEK machine
  - Client-side uses CPS

Source and target calculi

Our source calculus is $\lambda^{\rho}_{\text{eff}}$ (Hillerström and Lindley 2016).

$$
\begin{array}{lrl}
\text{Values} & V, W & ::= x \mid \lambda x.\, M \mid \langle\rangle \mid \langle \ell = V; W \rangle \mid (\ell\, V) \\
\text{Computations} & M, N & ::= V\, W \\
& & \mid\ \ \textbf{let}\ \langle \ell = x; y \rangle = V\ \textbf{in}\ N \mid \textbf{case}\ V \{\ell\, x \mapsto M; y \mapsto N\} \mid \textbf{absurd}\, V \\
& & \mid\ \ \textbf{return}\ V \mid \textbf{let}\ x \leftarrow M\ \textbf{in}\ N \\
& & \mid\ \ \textbf{do}\ \ell\ V \mid \textbf{handle}\ M\ \textbf{with}\ H \\
\text{Handlers} & H & ::= \{\textbf{return}\ x \mapsto M\} \mid \{\ell\ p\ r \mapsto M\} \uplus H\}
\end{array}
$$

Our target is an untyped λ-calculus

$$
\begin{array}{lrl}
\text{Values} & V, W & ::= x \mid \lambda x.M \mid \langle\rangle \mid \langle \ell = V; W \rangle \mid \ell\ V \\
\text{Computations} & M, N & ::= V \mid M\, W \mid \textbf{let}\ \langle \ell = x; y \rangle = V\ \textbf{in}\ N \\
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\end{array}
$$

Curried CPS translation for fine-grain call-by-value

The continuation represents the *execution context*.

Computations

$$
\begin{aligned}
[\![V\ W]\!] &= [\![V]\!]\ [\![W]\!] \\
[\![\mathbf{let}\ \langle \ell = x; y\rangle = V\ \mathbf{in}\ N]\!] &= \mathbf{let}\ \langle \ell = x; y\rangle = [\![V]\!]\ \mathbf{in}\ [\![N]\!] \\
[\![\mathbf{case}\ V\ \{\ell\ x \mapsto M; y \mapsto N\}]\!] &= \mathbf{case}\ [\![V]\!]\ \{\ell\ x \mapsto [\![M]\!]; y \mapsto [\![N]\!]\} \\
[\![\mathbf{absurd}\ V]\!] &= \mathbf{absurd}\ [\![V]\!] \\
[\![\mathbf{return}\ V]\!] &= \lambda k.k\ [\![V]\!] \\
[\![\mathbf{let}\ x \leftarrow M\ \mathbf{in}\ N]\!] &= \lambda k.[\![M]\!](\lambda x.[\![N]\!]k)
\end{aligned}
$$

Values

$$
\begin{aligned}
[\![x]\!] &= x \\
[\![\lambda x.M]\!] &= \lambda x.[\![M]\!] \\
[\![\langle\rangle]\!] &= \langle\rangle \\
[\![\langle \ell = V; W\rangle]\!] &= \langle \ell = [\![V]\!]; [\![W]\!]\rangle \\
[\![\ell\ V]\!] &= \ell\ [\![V]\!]
\end{aligned}
$$

Curried CPS for handlers

With handlers there are two contexts: a *pure* and an *effect* context.
Idea: implicit stack of alternating pure and effect continuations.

$$
\begin{aligned}
[\![\mathbf{do}\ \ell\ V]\!] &= \lambda k.\lambda h.h\ (\ell\ \langle [\![V]\!], \lambda x.k\ x\ h\rangle) \\
[\![\mathbf{handle}\ M\ \mathbf{with}\ H]\!] &= [\![M]\!]\ [\![H^{\mathrm{ret}}]\!]\ [\![H^{\mathrm{ops}}]\!] \\
[\![\{\mathbf{return}\ x \mapsto N\}]\!] &= \lambda x.\lambda h.[\![N]\!] \\
[\![\{\ell\ p\ r \mapsto N_\ell\}_{\ell \in \mathcal{L}}]\!] &= \lambda z.\mathbf{case}\ z\ \{(\ell\ \langle p, r\rangle \mapsto [\![N_\ell]\!])_{\ell \in \mathcal{L}}; y \mapsto M_{\mathrm{forward}}\} \\
M_{\mathrm{forward}} &= \lambda k'.\lambda h'.\mathbf{vmap}\,(\lambda \langle p, r\rangle\ k.k\ \langle p, \lambda x.r\,x\,k'\ h'\rangle)\ y\ h' \\
\top[\![M]\!] &= [\![M]\!]\ (\lambda x.\lambda h.x)\ (\lambda z.\mathbf{absurd}\ z)
\end{aligned}
$$

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Idea: implicit stack of alternating pure and effect continuations.

$$
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\boxed{[\![\mathbf{do}\ \ell\ V]\!] = \lambda k.\lambda h.h\ (\ell\ \langle [\![V]\!], \lambda x.k\ x\ h\rangle)} \\
[\![\mathbf{handle}\ M\ \mathbf{with}\ H]\!] &= [\![M]\!]\ [\![H^{\mathrm{ret}}]\!]\ [\![H^{\mathrm{ops}}]\!] \\
[\![\{\mathbf{return}\ x \mapsto N\}]\!] &= \lambda x.\lambda h.[\![N]\!] \\
[\![\{\ell\ p\ r \mapsto N_\ell\}_{\ell\in\mathcal{L}}]\!] &= \lambda z.\mathbf{case}\ z\ \{(\ell\ \langle p, r\rangle \mapsto [\![N_\ell]\!])_{\ell\in\mathcal{L}}; y \mapsto M_{\mathrm{forward}}\} \\
M_{\mathrm{forward}} &= \lambda k'.\lambda h'.\mathbf{vmap}\,(\lambda\langle p, r\rangle\ k.k\ \langle p, \lambda x.r\, x\, k'\ h'\rangle)\ y\ h' \\
\top[\![M]\!] &= [\![M]\!]\ (\lambda x.\lambda h.x)\ (\lambda z.\mathbf{absurd}\ z)
\end{aligned}
$$

Curried CPS for handlers

With handlers there are two contexts: a *pure* and an *effect* context.
Idea: implicit stack of alternating pure and effect continuations.

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M_{\mathrm{forward}} &= \lambda k'.\lambda h'.\mathbf{vmap}\,(\lambda\langle p, r\rangle\ k.k\ \langle p, \lambda x.r\,x\,k'\ h'\rangle)\ y\ h' \\
\boxed{\top[\![M]\!]} &\boxed{= [\![M]\!]\ (\lambda x.\lambda h.x)\ (\lambda z.\mathbf{absurd}\ z)}
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\end{aligned}
$$

Problem: $[\![-]\!]$ yields administrative redexes and is not properly tail-recursive!

$$
\top[\![\mathbf{return}\ \langle\rangle]\!] \ = \ (\lambda k.k\ \langle\rangle)\,(\lambda x.\lambda h.x)\,(\lambda z.\mathbf{absurd}\,z)
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$$
\begin{aligned}
\top[\![\mathbf{return}\ \langle\rangle]\!] &= (\lambda k.k\ \langle\rangle)\,(\lambda x.\lambda h.x)\,(\lambda z.\mathbf{absurd}\,z) \\
&\rightsquigarrow ((\lambda x.\lambda h.x)\ \langle\rangle)\,(\lambda z.\mathbf{absurd}\,z)
\end{aligned}
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Curried CPS for handlers

With handlers there are two contexts: a *pure* and an *effect* context.
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\end{aligned}
$$

Problem: $[\![-]\!]$ yields administrative redexes and is not properly tail-recursive!

$$
\begin{aligned}
\top[\![\mathbf{return}\ \langle\rangle]\!] \ &= \ (\lambda k.k\ \langle\rangle)\,(\lambda x.\lambda h.x)\,(\lambda z.\mathbf{absurd}\,z) \\
&\leadsto \ ((\lambda x.\lambda h.x)\ \langle\rangle)\ (\lambda z.\mathbf{absurd}\,z) \leadsto (\lambda h.\langle\rangle)\ (\lambda z.\mathbf{absurd}\,z)
\end{aligned}
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Curried CPS for handlers

With handlers there are two contexts: a *pure* and an *effect* context.
Idea: implicit stack of alternating pure and effect continuations.

$$\begin{aligned}
[\![\textbf{do}\ \ell\ V]\!] &= \lambda k.\lambda h.h\ (\ell\ \langle [\![V]\!], \lambda x.k\ x\ h\rangle) \\
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Problem: $[\![-]\!]$ yields administrative redexes and is not properly tail-recursive!

$$\begin{aligned}
\top[\![\textbf{return}\ \langle\rangle]\!] &= (\lambda k.k\ \langle\rangle)\,(\lambda x.\lambda h.x)\,(\lambda z.\textbf{absurd}\, z) \\
&\rightsquigarrow ((\lambda x.\lambda h.x)\ \langle\rangle)\ (\lambda z.\textbf{absurd}\, z) \rightsquigarrow (\lambda h.\langle\rangle)\ (\lambda z.\textbf{absurd}\, z) \rightsquigarrow \langle\rangle
\end{aligned}$$

Uncurried CPS for handlers

Idea: make the continuation stack explicit (Materzok and Biernacki 2012).

$$
\begin{aligned}
[\![\mathbf{return}\ V]\!] &= \lambda(k :: ks).k\ [\![V]\!]\ ks \\
[\![\mathbf{let}\ x \leftarrow M\ \mathbf{in}\ N]\!] &= \lambda(k :: ks).[\![M]\!]((\lambda x\, ks.[\![N]\!](k :: ks)) :: ks) \\
[\![\mathbf{do}\ \ell\ V]\!] &= \lambda(k :: h :: ks).h\,(\ell\ \langle [\![V]\!], h :: k :: []\rangle)\ ks \\
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M_{\mathrm{forward}} &= \mathbf{let}\ (k' :: h' :: ks') = ks\ \mathbf{in} \\
&\quad\ \mathbf{vmap}\,(\lambda\langle p, r\rangle\,(k :: ks).k\,\langle p, h' :: k' :: r\rangle\ ks)\ y\ ks' \\
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\end{aligned}
$$

But $[\![-]\!]$ still yields the administrative redexes

$$
\top[\![\mathbf{return}\ \langle\rangle]\!] \;=\; (\lambda(k :: ks).k\ \langle\rangle\ ks)\,((\lambda x\ \ ks.x) :: (\lambda z.\mathbf{absurd}\, z) :: [])
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$$
\begin{aligned}
\top[\![\mathbf{return}\ \langle\rangle]\!] &= (\lambda(k :: ks).k\ \langle\rangle\ ks)\ ((\lambda x\ ks.x) :: (\lambda z.\mathbf{absurd}\, z) :: []) \\
&\rightsquigarrow (\lambda x\ ks.x)\ \langle\rangle\ [(\lambda z.\mathbf{absurd}\, z)]
\end{aligned}
$$

Uncurried CPS for handlers

Idea: make the continuation stack explicit (Materzok and Biernacki 2012).

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[\![\mathbf{do}\ \ell\ V]\!] &= \lambda(k :: h :: ks).h\,(\ell\ \langle [\![V]\!], h :: k :: [] \rangle)\ ks \\
[\![\mathbf{handle}\ M\ \mathbf{with}\ H]\!] &= \lambda ks.[\![M]\!]([\![H^{\mathrm{ret}}]\!] :: [\![H^{\mathrm{ops}}]\!] :: ks) \\
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But $[\![-]\!]$ still yields the administrative redexes

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\begin{aligned}
\top[\![\mathbf{return}\ \langle\rangle]\!] &= (\lambda(k :: ks).k\ \langle\rangle\ ks)\,((\lambda x\ \ ks.x) :: (\lambda z.\mathbf{absurd}\, z) :: []) \\
&\rightsquigarrow (\lambda x\ \ ks.x)\ \langle\rangle\ [(\lambda z.\mathbf{absurd}\, z)] \rightsquigarrow^{+} \langle\rangle
\end{aligned}
$$

Higher-order CPS for handlers (I)

Idea: adopt a two level λ-calculus (Danvy and Filinski 1990; Danvy and Nielsen 2003) with two kinds of continuations

- Static continuations κ
- Dynamic continuations k

Correspondingly, two kinds of reductions

- $\overline{\textit{Static}}$ reductions at translation time
- $\underline{\textit{Dynamic}}$ reductions at run-time

Move between levels using reflection ($\uparrow$) and reification ($\downarrow$)

$$\begin{aligned} \downarrow\uparrow V &= V \\ \downarrow(V \mathbin{\overline{::}} VS) &= V \mathbin{\overline{::}} \downarrow VS \end{aligned}$$

Higher-order CPS for handlers (II)

Two kinds of reductions: $\overline{\textit{static}}$ and $\underline{\textit{dynamic}}$.

$$
\begin{aligned}
[\![\textbf{return } V]\!] &= \overline{\lambda}\kappa \mathbin{\overline{::}} \kappa s.\kappa \mathbin{\underline{@}} [\![V]\!] \mathbin{\underline{@}} {\downarrow}\kappa s \\
[\![\textbf{let } x \leftarrow M \textbf{ in } N]\!] &= \overline{\lambda}\kappa \mathbin{\overline{::}} \kappa s.[\![M]\!] \mathbin{\overline{@}} ((\underline{\lambda} x\, ks.[\![N]\!] \mathbin{\overline{@}} (\kappa \mathbin{\overline{::}} {\uparrow}ks)) \mathbin{\overline{::}} \kappa s) \\
[\![\textbf{do } \ell\ V]\!] &= \overline{\lambda}\kappa \mathbin{\overline{::}} \eta \mathbin{\overline{::}} \kappa s.\eta \mathbin{\underline{@}} (\ell\ \langle [\![V]\!], \eta \mathbin{\underline{::}} \kappa \mathbin{\underline{::}} [\,]\rangle) \mathbin{\underline{@}} {\downarrow}\kappa s \\
[\![\textbf{handle } M \textbf{ with } H]\!] &= \overline{\lambda}\kappa s.[\![M]\!] \mathbin{\overline{@}} ([\![H^{\mathrm{ret}}]\!] \mathbin{\overline{::}} [\![H^{\mathrm{ops}}]\!] \mathbin{\overline{::}} \kappa s) \\
[\![\{\textbf{return } x \mapsto N\}]\!] &= \underline{\lambda} x\, ks.\underline{\textbf{let}}\ (h \mathbin{\underline{::}} ks') = ks \mathbin{\underline{\textbf{in}}} [\![N]\!] \mathbin{\overline{@}} {\uparrow}ks' \\
[\![\{(\ell\ p\ r \mapsto N_\ell)_{\ell \in \mathcal{L}}\}]\!] &= \underline{\lambda} z\, ks.\textbf{case } z\ \{(\ell\ \langle p, s\rangle \mapsto \underline{\textbf{let}}\ r = \textbf{fun}\, s \mathbin{\underline{\textbf{in}}} [\![N_\ell]\!] \mathbin{\overline{@}} {\uparrow}ks)_{\ell \in \mathcal{L}}; \\
&\qquad\qquad\qquad y \mapsto M_{\mathrm{forward}}\} \\
M_{\mathrm{forward}} &= \underline{\textbf{let}}\ (k' \mathbin{\underline{::}} h' \mathbin{\underline{::}} ks') = ks \mathbin{\underline{\textbf{in}}} \\
&\ \textbf{vmap}\, (\underline{\lambda}\langle p, s\rangle\ (k \mathbin{\underline{::}} ks).k\ \langle p, h' \mathbin{\underline{::}} k' \mathbin{\underline{::}} s\rangle\ ks)\ y\ ks' \\
\top[\![M]\!] &= [\![M]\!] \mathbin{\overline{@}} ((\underline{\lambda} x\, ks.x) \mathbin{\overline{::}} (\underline{\lambda} z\, ks.\textbf{absurd } z) \mathbin{\overline{::}} {\uparrow}[\,])
\end{aligned}
$$

Higher-order CPS for handlers (II)

Two kinds of reductions: $\overline{\textit{static}}$ and $\underline{\textit{dynamic}}$.

$$
\begin{aligned}
[\![\mathbf{return}\ V]\!] &= \overline{\lambda}\kappa \mathbin{\overline{::}} \kappa s.\kappa \mathbin{\underline{@}} [\![V]\!] \mathbin{\underline{@}} {\downarrow}\kappa s \\
[\![\mathbf{let}\ x \leftarrow M\ \mathbf{in}\ N]\!] &= \overline{\lambda}\kappa \mathbin{\overline{::}} \kappa s.[\![M]\!] \mathbin{\overline{@}} ((\underline{\lambda} x\, ks.[\![N]\!] \mathbin{\overline{@}} (\kappa \mathbin{\overline{::}} {\uparrow} ks)) \mathbin{\overline{::}} \kappa s) \\
[\![\mathbf{do}\ \ell\ V]\!] &= \overline{\lambda}\kappa \mathbin{\overline{::}} \eta \mathbin{\overline{::}} \kappa s.\eta \mathbin{\underline{@}} (\ell\ \langle [\![V]\!], \eta \mathbin{\underline{::}} \kappa \mathbin{\underline{::}} []\rangle) \mathbin{\underline{@}} {\downarrow}\kappa s \\
[\![\mathbf{handle}\ M\ \mathbf{with}\ H]\!] &= \overline{\lambda}\kappa s.[\![M]\!] \mathbin{\overline{@}} ([\![H^{\mathrm{ret}}]\!] \mathbin{\overline{::}} [\![H^{\mathrm{ops}}]\!] \mathbin{\overline{::}} \kappa s) \\
[\![\{\mathbf{return}\ x \mapsto N\}]\!] &= \underline{\lambda} x\, ks.\underline{\mathbf{let}}\ (h \mathbin{\underline{::}} ks') = ks\ \underline{\mathbf{in}}\ [\![N]\!] \mathbin{\overline{@}} {\uparrow} ks' \\
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\end{aligned}
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Two kinds of reductions: $\overline{\textit{static}}$ and $\underline{\textit{dynamic}}$.

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\end{aligned}
$$

$[\![-]\!]$ produces no statically eliminatable administrative redexes

$$
\top[\![\mathbf{return}\ \langle\rangle]\!] \;=\; (\overline{\lambda}\kappa \mathbin{\overline{::}} \kappa s.\kappa \mathbin{\underline{@}} \langle\rangle \mathbin{\underline{@}} {\downarrow}\kappa) \mathbin{\overline{@}} ((\underline{\lambda} x\ ks.x) \mathbin{\overline{::}} (\underline{\lambda} z\ ks.\mathbf{absurd}\ z) \mathbin{\overline{::}} {\uparrow}[\,])
$$

Higher-order CPS for handlers (II)

Two kinds of reductions: $\overline{\textit{static}}$ and $\underline{\textit{dynamic}}$.

$$
\begin{aligned}
\llbracket \mathbf{return}\ V \rrbracket &= \overline{\lambda}\kappa \mathbin{\overline{::}} \kappa s.\kappa \mathbin{\underline{@}} \llbracket V \rrbracket \mathbin{\underline{@}} {\downarrow}\kappa s \\
\llbracket \mathbf{let}\ x \leftarrow M\ \mathbf{in}\ N \rrbracket &= \overline{\lambda}\kappa \mathbin{\overline{::}} \kappa s.\llbracket M \rrbracket \mathbin{\overline{@}} ((\underline{\lambda} x\, ks.\llbracket N \rrbracket \mathbin{\overline{@}} (\kappa \mathbin{\overline{::}} {\uparrow}ks)) \mathbin{\overline{::}} \kappa s) \\
\llbracket \mathbf{do}\ \ell\ V \rrbracket &= \overline{\lambda}\kappa \mathbin{\overline{::}} \eta \mathbin{\overline{::}} \kappa s.\eta \mathbin{\underline{@}} (\ell\ \langle \llbracket V \rrbracket, \eta \mathbin{\underline{::}} \kappa \mathbin{\underline{::}} [\,]\rangle) \mathbin{\underline{@}} {\downarrow}\kappa s \\
\llbracket \mathbf{handle}\ M\ \mathbf{with}\ H \rrbracket &= \overline{\lambda}\kappa s.\llbracket M \rrbracket \mathbin{\overline{@}} (\llbracket H^{\mathrm{ret}} \rrbracket \mathbin{\overline{::}} \llbracket H^{\mathrm{ops}} \rrbracket \mathbin{\overline{::}} \kappa s) \\
\llbracket \{\mathbf{return}\ x \mapsto N\} \rrbracket &= \underline{\lambda} x\, ks.\underline{\mathbf{let}}\ (h \mathbin{\underline{::}} ks') = ks\ \underline{\mathbf{in}}\ \llbracket N \rrbracket \mathbin{\overline{@}} {\uparrow}ks' \\
\llbracket \{(\ell\ p\ r \mapsto N_\ell)_{\ell \in \mathcal{L}}\} \rrbracket &= \underline{\lambda} z\, ks.\mathbf{case}\ z\ \{(\ell\ \langle p, s\rangle \mapsto \underline{\mathbf{let}}\ r = \mathbf{fun}\ s\ \underline{\mathbf{in}}\ \llbracket N_\ell \rrbracket \mathbin{\overline{@}} {\uparrow}ks)_{\ell \in \mathcal{L}}; \\
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M_{\mathrm{forward}} &= \underline{\mathbf{let}}\ (k' \mathbin{\underline{::}} h' \mathbin{\underline{::}} ks') = ks\ \underline{\mathbf{in}} \\
&\ \mathbf{vmap}\ (\underline{\lambda}\langle p, s\rangle\ (k \mathbin{\underline{::}} ks).k\ \langle p, h' \mathbin{\underline{::}} k' \mathbin{\underline{::}} s\rangle\ ks)\ y\ ks' \\
\top\llbracket M \rrbracket &= \llbracket M \rrbracket \mathbin{\overline{@}} ((\underline{\lambda} x\, ks.x) \mathbin{\overline{::}} (\underline{\lambda} z\, ks.\mathbf{absurd}\ z) \mathbin{\overline{::}} {\uparrow}[\,])
\end{aligned}
$$

$\llbracket - \rrbracket$ produces no statically eliminatable administrative redexes

$$
\begin{aligned}
\top\llbracket \mathbf{return}\ \langle\rangle \rrbracket &= (\overline{\lambda}\kappa \mathbin{\overline{::}} \kappa s.\kappa \mathbin{\underline{@}} \langle\rangle \mathbin{\underline{@}} {\downarrow}\kappa) \mathbin{\overline{@}} ((\underline{\lambda} x\, ks.x) \mathbin{\overline{::}} (\underline{\lambda} z\, ks.\mathbf{absurd}\ z) \mathbin{\overline{::}} {\uparrow}[\,]) \\
&= (\underline{\lambda} x\, ks.x) \mathbin{\underline{@}} \langle\rangle \mathbin{\underline{@}} {\downarrow}((\underline{\lambda} z\, ks.\mathbf{absurd}\ z) \mathbin{\overline{::}} {\uparrow}[\,])
\end{aligned}
$$

Higher-order CPS for handlers (II)

Two kinds of reductions: $\overline{\textit{static}}$ and $\underline{\textit{dynamic}}$.

$$
\begin{aligned}
[\![\mathbf{return}\ V]\!] &= \overline{\lambda}\kappa \mathbin{\overline{::}} \kappa s.\kappa \mathbin{\underline{@}} [\![V]\!] \mathbin{\underline{@}} {\downarrow}\kappa s \\
[\![\mathbf{let}\ x \leftarrow M\ \mathbf{in}\ N]\!] &= \overline{\lambda}\kappa \mathbin{\overline{::}} \kappa s.[\![M]\!] \mathbin{\overline{@}} ((\underline{\lambda} x\, ks.[\![N]\!] \mathbin{\overline{@}} (\kappa \mathbin{\overline{::}} {\uparrow}ks)) \mathbin{\overline{::}} \kappa s) \\
[\![\mathbf{do}\ \ell\ V]\!] &= \overline{\lambda}\kappa \mathbin{\overline{::}} \eta \mathbin{\overline{::}} \kappa s.\eta \mathbin{\underline{@}} (\ell\ \langle [\![V]\!], \eta \mathbin{\underline{::}} \kappa \mathbin{\underline{::}} [\,]\rangle) \mathbin{\underline{@}} {\downarrow}\kappa s \\
[\![\mathbf{handle}\ M\ \mathbf{with}\ H]\!] &= \overline{\lambda}\kappa s.[\![M]\!] \mathbin{\overline{@}} ([\![H^{\mathrm{ret}}]\!] \mathbin{\overline{::}} [\![H^{\mathrm{ops}}]\!] \mathbin{\overline{::}} \kappa s) \\
[\![\{\mathbf{return}\ x \mapsto N\}]\!] &= \underline{\lambda} x\, ks.\underline{\mathbf{let}}\ (h \mathbin{\underline{::}} ks') = ks\ \underline{\mathbf{in}}\ [\![N]\!] \mathbin{\overline{@}} {\uparrow}ks' \\
[\![\{(\ell\ p\ r \mapsto N_\ell)_{\ell \in \mathcal{L}}\}]\!] &= \underline{\lambda} z\, ks.\mathbf{case}\ z\ \{(\ell\ \langle p, s\rangle \mapsto \underline{\mathbf{let}}\ r = \mathbf{fun}\, s\ \underline{\mathbf{in}}\ [\![N_\ell]\!] \mathbin{\overline{@}} {\uparrow}ks)_{\ell \in \mathcal{L}}; \\
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&\quad\ \mathbf{vmap}\ (\underline{\lambda}\langle p, s\rangle\ (k \mathbin{\underline{::}} ks).k\ \langle p, h' \mathbin{\underline{::}} k' \mathbin{\underline{::}} s\rangle\ ks)\ y\ ks' \\
\top[\![M]\!] &= [\![M]\!] \mathbin{\overline{@}} ((\underline{\lambda} x\, ks.x) \mathbin{\overline{::}} (\underline{\lambda} z\, ks.\mathbf{absurd}\ z) \mathbin{\overline{::}} {\uparrow}[\,])
\end{aligned}
$$

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$$
\begin{aligned}
\top[\![\mathbf{return}\ \langle\rangle]\!] &= (\overline{\lambda}\kappa \mathbin{\overline{::}} \kappa s.\kappa \mathbin{\underline{@}} \langle\rangle \mathbin{\underline{@}} {\downarrow}\kappa) \mathbin{\overline{@}} ((\underline{\lambda} x\, ks.x) \mathbin{\overline{::}} (\underline{\lambda} z\, ks.\mathbf{absurd}\, z) \mathbin{\overline{::}} {\uparrow}[\,]) \\
&= (\underline{\lambda} x\, ks.x) \mathbin{\underline{@}} \langle\rangle \mathbin{\underline{@}} {\downarrow}((\underline{\lambda} z\, ks.\mathbf{absurd}\, z) \mathbin{\overline{::}} {\uparrow}[\,]) \\
&= (\underline{\lambda} x\, ks.x) \mathbin{\underline{@}} \langle\rangle \mathbin{\underline{@}} ((\underline{\lambda} z\, ks.\mathbf{absurd}\, z) \mathbin{\overline{::}} [\,])
\end{aligned}
$$

Higher-order CPS for handlers (II)

Two kinds of reductions: $\overline{\textit{static}}$ and $\underline{\textit{dynamic}}$.

$$
\begin{aligned}
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[\![\textbf{handle } M \textbf{ with } H]\!] &= \overline{\lambda}\kappa s.[\![M]\!] \mathbin{\overline{@}} ([\![H^{\mathrm{ret}}]\!] \mathbin{\overline{::}} [\![H^{\mathrm{ops}}]\!] \mathbin{\overline{::}} \kappa s) \\
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\top[\![M]\!] &= [\![M]\!] \mathbin{\overline{@}} ((\underline{\lambda} x\, ks.x) \mathbin{\overline{::}} (\underline{\lambda} z\, ks.\textbf{absurd } z) \mathbin{\overline{::}} \uparrow[])
\end{aligned}
$$

$[\![-]\!]$ produces no statically eliminatable administrative redexes

$$
\begin{aligned}
\top[\![\textbf{return } \langle\rangle]\!] &= (\overline{\lambda}\kappa \mathbin{\overline{::}} \kappa s.\kappa \mathbin{\underline{@}} \langle\rangle \mathbin{\underline{@}} \downarrow\kappa) \mathbin{\overline{@}} ((\underline{\lambda} x\, ks.x) \mathbin{\overline{::}} (\underline{\lambda} z\, ks.\textbf{absurd } z) \mathbin{\overline{::}} \uparrow[]) \\
&= (\underline{\lambda} x\, ks.x) \mathbin{\underline{@}} \langle\rangle \mathbin{\underline{@}} \downarrow((\underline{\lambda} z\, ks.\textbf{absurd } z) \mathbin{\overline{::}} \uparrow[]) \\
&= (\underline{\lambda} x\, ks.x) \mathbin{\underline{@}} \langle\rangle \mathbin{\underline{@}} ((\underline{\lambda} z\, ks.\textbf{absurd } z) \mathbin{\overline{::}} []) \rightsquigarrow^{+} \langle\rangle
\end{aligned}
$$

Correctness

Definition (Translation of evaluation contexts)

$$
\begin{aligned}
[\![[\,]]\!] &= \overline{\lambda}\kappa s.\ \kappa s \\
[\![\mathbf{let}\ x \leftarrow \mathcal{E}\ \mathbf{in}\ N]\!] &= \overline{\lambda}\kappa \mathbin{\overline{::}} \kappa s.\ [\![\mathcal{E}]\!] \mathbin{\overline{@}} ((\underline{\lambda} x\, ks.[\![N]\!] \mathbin{\overline{@}} (k \mathbin{\overline{::}} \uparrow ks)) \mathbin{\overline{::}} \kappa s) \\
[\![\mathbf{handle}\ \mathcal{E}\ \mathbf{with}\ H]\!] &= \overline{\lambda}\kappa s.\ [\![\mathcal{E}]\!] \mathbin{\overline{@}} ([\![H^{\mathrm{ret}}]\!] \mathbin{\overline{::}} [\![H^{\mathrm{ops}}]\!] \mathbin{\overline{::}} \kappa s)
\end{aligned}
$$

Lemma (Decomposition)

The characteristic property of the CPS translation on evaluation contexts

$$[\![\mathcal{E}[M]]\!] \mathbin{\overline{@}} (V \mathbin{\overline{::}} VS) = [\![M]\!] \mathbin{\overline{@}} ([\![\mathcal{E}]\!] \mathbin{\overline{@}} (V \mathbin{\overline{::}} VS))$$

Theorem (Simulation)

If $M \rightsquigarrow N$ *then* $\top[\![M]\!] \rightsquigarrow^{+} \rightsquigarrow_a^{*} \top[\![N]\!]$.

Conclusions

In summary

- handlers provide a modular abstraction for user-defined computational effects,
- first full CPS translations for handlers,
- the first-order curried CPS is neat, but inpractical,
- and the first-order uncurried CPS is naïve,
- refectified by a higher-order uncurried CPS.

Full details in the paper along with

- a discussion on the implementation of the higher-order CPS,
- a sketch of a typed higher-order CPS translation,
- an adaptation to accommodate *shallow* handlers,
- and a discussion on adapting the translation to a language like OCaml.

An implementation is available in Links

`https://github.com/links-lang/links`

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